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Exploring an S-matrix for gravitational collapse

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Abstract: We analyze further a recently proposed S-matrix description of transplanckian scattering in the specific case of axisymmetric collisions of extended sources, where some of the original approximations are not necessary. We confirm the claim that such an approximate description appears to capture the essential features of (the quantum counterpart of) classical gravitational collapse. More specifically, the S-matrix develops singularities (branch points corresponding to some new absorption of the elastic amplitude) whose location in the sources' parameter space are consistent with (and numerically close to) the bounds coming from closed-trapped-surface collapse criteria. In the vicinity of the critical "lines" the phase of the elastic S-matrix exhibits a universal fractional-power behaviour reminiscent of Choptuik's scaling near critical collapse.

Keywords: Models of Quantum Gravity, Black Holes, Spacetime Singularities.

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Contents

1. Introduction

In a recent paper [1] (hereafter referred to as ACV) a simplified non-perturbative treatment of transplanckian scattering has been proposed. It is based on an approximate resummation of the semi-classical corrections to the leading eikonal approximation. These were identified long ago [2] as coming from tree diagrams where the fast initial particles act as external sources for gravitons.

Resumming tree diagrams amounts to solving the equations of motion of an effective action proposed quite sometime ago [3, 4]. Taking the high-energy limit simplifies considerably the longitudinal dynamics which, in a certain approximation (neglect of so-called "rescattering diagrams"), can be factored out leaving behind an effective dynamics in the "transverse" $(D-2)$-dimensional space. In [1] (and here) the case of $D = 4$, hence of an effective *two*-dimensional theory, was (will be) considered.

In [1] one further simplification consisted of neglecting a physical polarization of the radiated gravitons, the one suffering from infrared problems. Finally, some azimuthal averaging was used in order to be able to treat the problem both analytically and numerically. In spite of these approximations, the resulting S-matrix looked able to capture the essential physics of the gravitational collapse problem. In particular, it showed the existence of a critical impact parameter b_c below which a new absorption (and thus the opening of some new channels) occurs. That critical value, possibly signalling a transition between what we may call a dispersive (D) and a black-hole (BH) phase, turned out to be in good agreement with the one based on closed-trapped-surface (CTS) criteria [5, 6]. A more recent investigation [7] managed to solve numerically the full partial differential equations (PDE) (i.e. without using the azimuthal average approximation) and largely confirmed the qualitative outcome of the simplified ACV approach.

In this paper we consider more systematically the case of axisymmetric extended colliding sources/beams. The motivations for considering this particular case are several: on the one hand the PDE's reduce to ODE's making the problem an affordable one by analytic (or by much easier numerical) techniques, without having to make the azimuthal averaging approximation. At the same time, as we will show, the IR-sensitive polarization is simply not produced in the axisymmetic case. Last but not least, we are able to introduce a vaste number of initial states, by playing at will with the many (shape and intensity) parameters chracterizing the sources and check for the existence of critical surfaces in this multidimensional space. The results can then be compared with those coming from CTS criteria [6] and, hopefully in the near future, will be tested against analytic [8] as well as numerical GR calculations [9].

The paper is organized as follows: in section 2 we make some general considerations on the axisymmetric case and prove a one-way relation between the CTS criterion of [6] and the criticality condition in the ACV system. In section 3 we give some examples of interesting extended sources while in section 4 we determine numerically the critical lines in the sources' parameter space. We also compare those lines with the bounds coming from the (much simpler) CTS conditions and find very good agreement. In section 5 we study the behaviour of the elastic S-matrix in the vicinity of the critical points and point out a possible connection with Choptuik's scaling [10] near critical collapse. Section 6 briefly summarizes our results and future prospects.

2. The axisymmetric case: general considerations

Our starting point is the effective two-dimensional action of [1] (see their equation (2.22)):

$$\begin{aligned} \frac{\mathcal{A}}{2\pi G s} &= a(\boldsymbol{b}) + \bar{a}(0) - \frac{1}{2}\int d^2\boldsymbol{x} \nabla\bar{a}\nabla a + \frac{(\pi R)^2}{2}\int d^2\boldsymbol{x}(-(\nabla^2\phi)^2 + 2\mathcal{H}\nabla^2\phi)\,, \\ -\nabla^2\mathcal{H} &\equiv \nabla^2 a\ \nabla^2\bar{a} - \nabla_i\nabla_j a\ \nabla_i\nabla_j\bar{a}\,, \end{aligned} \tag{2.1}$$

where a, $\bar{a}$ and ϕ are three real fields representing the two longitudinal and the (IR-safe) transverse component of the gravitational field, respectively. Equation (2.1) can be easily generalized in order to deal with two extended sources:

$$\begin{aligned} \frac{\mathcal{A}}{2\pi G s} = \int d^2x \left[a(x)\bar{s}(x) + \bar{a}(x)s(x) - \frac{1}{2}\nabla_i\bar{a}\nabla_i a\right] \\ + \frac{(\pi R)^2}{2}\int d^2x\left(-(\nabla^2\phi)^2 + 2\phi\nabla^2\mathcal{H}\right)\,, \end{aligned} \tag{2.2}$$

where the center of mass energy $\sqrt{s}$ provides the overall normalization factor $2\pi Gs = \frac{\pi}{2G}R^2$, while the two sources $s(x), \bar{s}(x)$ are normalized by $\int d^2x\ s(x) = \int d^2x\ \bar{s}(x) = 1$.

Let us now specialize to the case of two extended axisymmetric sources moving in opposite direction with the speed of light and undergoing a central collision. Using the conventions of [1] we will denote by $E_i(r_i)$ (in the following $i = 1, 2$ will represent unbarred and barred fields/sources respectively) the energy carried by the ith beam below $r = r_i$ and define $R_i(r) = 4GE_i(r)$. Let us also assume that the two sources have finite support so that $R_i(r) = R_i(\infty) \equiv R_i$ for $r > L_i$. By going to the overall center of mass, we may always choose $R_i = R = 2G\sqrt{s}$.

2.1 Simplifications

One advantage of considering the axisymmetric case is that there is simply no dependence of the physics upon the azimuthal angle, hence no need to take averages over it. This is a useful technical simplification that allows us to reduce the problem to solving ODE.

The second more important advantage comes from the observation that the IR-singular "LT" graviton polarization is not produced in that case. Thus the problem is completely IR-finite even in $D = 4$. In order to see this, let us recall from [1] that the LT polarization is produced with an amplitude proportional to $\sin\theta_{12}\cos\theta_{12}$. In the notations of [1]:

$$A_{LT} = A_{\mu\nu}\epsilon^{\mu\nu}_{LT} \sim \boldsymbol{k}^{-2}\sin\theta_{12}\cos\theta_{12}\,, \tag{2.3}$$

where θ_{12} is the angle between the two transverse momenta $\boldsymbol{k}_1, \boldsymbol{k}_2$ that combine to give a physical graviton of momentum $\boldsymbol{k}$. The angular factor can be expressed in terms of the momenta as:

$$\boldsymbol{k}_1^2\boldsymbol{k}_2^2\sin\theta_{12}\cos\theta_{12} = \epsilon_{ij}\boldsymbol{k}_1^i\boldsymbol{k}_1^k\boldsymbol{k}_2^j\boldsymbol{k}_2^k\,. \tag{2.4}$$

Going over to position space this becomes a differential operator acting on a and $\bar{a}$:

$$\epsilon_{ij}\nabla^i\nabla^k a\nabla^j\nabla^k\bar{a}\,. \tag{2.5}$$

Given that in the axisymmetric case a and $\bar{a}$ depend only on $\boldsymbol{x}^2$ it is easy to see that the above expression vanishes as a result of the antisymmetry of ϵ_{ij}.

2.2 Axisymmetric action and equations of motion

It is straightforward to rewrite the action (2.2) for the axisymmetric case as a one dimensional integral over the variable $r^2 = x^2 \equiv t$. Using $\int d^2x = \pi\int_0^\infty dt$ we find:

$$\frac{\mathcal{A}}{2\pi^2 Gs} = \int dt\,[a(t)\bar{s}(t) + \bar{a}(t)s(t) - 2\rho\dot{\bar{a}}\dot{a}] - \frac{2}{(2\pi R)^2}\int dt(1-\dot{\rho})^2\,, \tag{2.6}$$

where a dot means d/dt and, as in [1], we have introduced the field:

$$\rho = t\left(1 - (2\pi R)^2\dot{\phi}\right)\,. \tag{2.7}$$

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Integrating by parts and using $\pi \int_0^t dt' s_i(t') = R_i(t)/R$ we arrive at the following convenient form of the action:

$$\frac{\mathcal{A}}{\hbar} = -\frac{1}{4l_P^2}\int dt \left[(1-\dot{\rho})^2 - \frac{1}{\rho}R_1(t)R_2(t) + (2\pi R)^2\rho\left(\dot{a}_1 + \frac{R_1(t)}{2\pi R\rho}\right)\left(\dot{a}_2 + \frac{R_2(t)}{2\pi R\rho}\right)\right]. \tag{2.8}$$

The equations of motion that follow from (2.8) read:

$$\begin{aligned} \dot{a}_i &= -\frac{1}{2\pi\rho}\frac{R_i(t)}{R}, \\ \ddot{\rho} &= \frac{1}{2}(2\pi R)^2\dot{a}_1\dot{a}_2 = \frac{1}{2}\frac{R_1(t)R_2(t)}{\rho^2}, \end{aligned} \tag{2.9}$$

and therefore reduce to a closed 2nd order equation for ρ. We want to look for solutions of that equation with the following boundary conditions [1]:

$$\rho(0) = 0 \ , \quad \rho(t) \to t \ \text{ as } \ t \to \infty\,. \tag{2.10}$$

Given the finite support of the sources the latter condition can be replaced by the requirement:

$$\dot{\rho} = \sqrt{1 - R^2/\rho} \quad \text{for} \quad \sqrt{t} > Max(L_1, L_2).. \tag{2.11}$$

For given source profiles $R_i(t)$ a possible strategy for solving the problem is to reduce it to a first order system:

$$\begin{aligned} \dot{\rho} &= \sqrt{\sigma - \frac{R_1(t)R_2(t)}{\rho}} \quad i.e. \quad \sigma \equiv \dot{\rho}^2 + \frac{R_1(t)R_2(t)}{\rho}, \\ \dot{\sigma} &= \frac{1}{\rho}\frac{d(R_1R_2)}{dt}, \end{aligned} \tag{2.12}$$

with initial conditions

$$\rho(0) = 0 \ , \quad \sigma(0) = \sigma_0, \ , \tag{2.13}$$

and to find a σ_0 such that $\sigma(Max(L_1, L_2)) = 1$. For sufficiently small (large) R_i/L_i one expects that the latter condition can (cannot) be imposed on real-valued solutions. Thus, in general, there should be some critical values for those parameters separating two different regimes.

2.3 CTS-criteria and critical points: a general result

In the general axisymmetric case, one can construct explicitly a minimal CTS [6] provided that an r_c exists such that (see eq. (4.4) of [6] for $D = 4$):

$$R_1(r_c)R_2(r_c) = r_c^2\,. \tag{2.14}$$

We will now argue that such a condition implies the absence of real solutions to eqs. (2.9) with $\rho(0) = 0$. Proof: Let us first note that, because of (2.12) and the fact that the R_i are non-decreasing functions of r, the quantity σ, as well as $\dot{\rho}$, are increasing functions of t. Therefore, for any t:

$$\sigma(t) \leq \sigma(\infty) = 1 \ , \quad \text{i.e.} \ \ \dot{\rho}(t) \leq \sqrt{1 - \frac{R_1(t)R_2(t)}{\rho(t)}} \,. \tag{2.15}$$

Assuming that the KV criterion (2.14) can be met let us write:

$$\rho(0) = \rho(t_c) - \int_0^{t_c} dt' \dot{\rho}(t') > \rho(t_c) - t_c\dot{\rho}(t_c) > \rho(t_c) - t_c\sqrt{1 - \frac{t_c}{\rho(t_c)}} \,, \tag{2.16}$$

where we have used eqs. (2.14) and (2.15). At this point it is easy to check that the rhs of (2.16) cannot vanish for any (positive) value of $\rho(t_c)$ thus proving that we cannot impose the condition $\rho(0) = 0$ when the criterion (2.14) is satisfied.

3. Examples of source profiles

3.1 Two identical two-parameter sources

A class of finite-size sources is characterized by the following profiles:

$$s_1(t) = s_2(t) = \frac{d}{\pi \left(d + (1-d)t^2\right)^{3/2}} \Theta(1-t) \,, \tag{3.1}$$

where, without lack of generality, we have fixed the sizes of the two beams to be 1. One can easily verify that these sources satisfy:

$$\pi \int_0^t dt' s(t') = R(t)/R = \frac{t}{\left(d + (1-d)t^2\right)^{1/2}} \ , \quad \pi \int_0^1 s(t) dt = 1 \,. \tag{3.2}$$

Inserting these expressions in our differential equation (2.12) leads to the equations:

$$\begin{aligned} \dot{\rho} &= \sqrt{\sigma - \frac{R^2(t)}{\rho}} \quad , \quad \dot{\sigma} = \frac{1}{\rho}\frac{dR^2}{dt} \,, \\ R^2(t) &= \frac{R^2 t^2}{d + (1-d)t^2} \quad , \quad t < 1 \,, \end{aligned} \tag{3.3}$$

while for $t > 1$ we should impose $\sigma = 1$.

This system can be easily studied numerically. In particular one can find how the critical value of R, R_c, depends on the parameter d. This parameter gives (for fixed total extent $L \equiv 1$) the "shape" of the extended sources: $d = 1$ corresponds to the case of constant-density sources considered in [1] , $d < 1$ to sources roughly concentrated around $r \sim \sqrt{d}L$, and $d > 1$ to sources peaked around $r = L = 1$.

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3.2 Central scattering of a particle off a ring

In this, very asymmetric case $R_1 = R$ while $R_2 = R\Theta(t - b^2)$. The notation anticipates that we can view this case as an approximation to the scattering of two particles at impact parameter b. And indeed, from (2.9), we recover in this case the approximation used in [1] to describe the latter process i.e.

$$\ddot{\rho} = \frac{1}{2}\frac{R_1(t)R_2(t)}{\rho^2} = \frac{1}{2}\frac{R^2}{\rho^2}\Theta(t - b^2)\,. \tag{3.4}$$

If we require $\rho(0) = 0$ this equation leads to the condition on $\rho(b^2)$:

$$\rho(b^2) = b^2\dot{\rho}(b^2) = b^2\sqrt{1 - \frac{R^2}{\rho(b)}}\,, \tag{3.5}$$

which has a real solution only if $\frac{R}{b} < (\frac{R}{b})_c = 2^{1/2}3^{-3/4} \sim 0.62$. Such a result has to be compared with the upper limit given by (2.14) which, in this case, simply becomes $(R/b)_c^{\rm CTS} < 1$.

It is interesting to notice that exactly the same $(\frac{R}{b})_c$ will apply to the situation in which the point-like source is replaced by an arbitrary source "contained" inside the ring-shaped one. Physically this makes sense since, by Gauss' theorem, the compact source should propagate undisturbed while the more extended source is only affected by the total energy of the more compact one.

3.3 Gaussian sources

Point-like sources are difficult to deal with numerically (especially in momentum space). Therefore we also introduce Gaussian-smeared versions of the above point and ring-like sources.

$$\begin{aligned}
s_1(\boldsymbol{x}) &= \frac{1}{\mathcal{N}_1}\exp\left(-\frac{r^2}{2\sigma^2}\right)\Theta(L_1 - r)\,,\\
,\, s_2(\boldsymbol{x}) &= \frac{1}{\mathcal{N}_2}\exp\left(-\frac{(r-L_2)^2}{2\sigma^2}\right)\Theta(L_2 - r),\\
\mathcal{N}_1 &= 2\pi\sigma^2\left(1 - \exp\left(\frac{-L_1^2}{2\sigma^2}\right)\right)\,, &&(3.6)\\
\mathcal{N}_2 &= 2\pi\left(\sigma^2(\exp\left(\frac{-L_2^2}{2\sigma^2}\right) - 1) + \sigma L_2\sqrt{\frac{\pi}{2}}\mathrm{Erf}\frac{\mathrm{L}_2}{\sqrt{2}\sigma}\right)\,. &&(3.7)
\end{aligned}$$

When $\sigma \longrightarrow 0$ (∞) the problem reduces to the one of the point-ring (two homogeneous beams) case.

Another interesting example is the one of gaussian sources concentrated at $r = 0$. They correspond to:

$$s_i(t) = \frac{1}{2\pi L_i^2}exp\left(-\frac{t}{2L_i^2}\right)\ ,\quad \frac{R_i(t)}{R} = 1 - exp\left(-\frac{t}{2L_i^2}\right)\,. \tag{3.8}$$

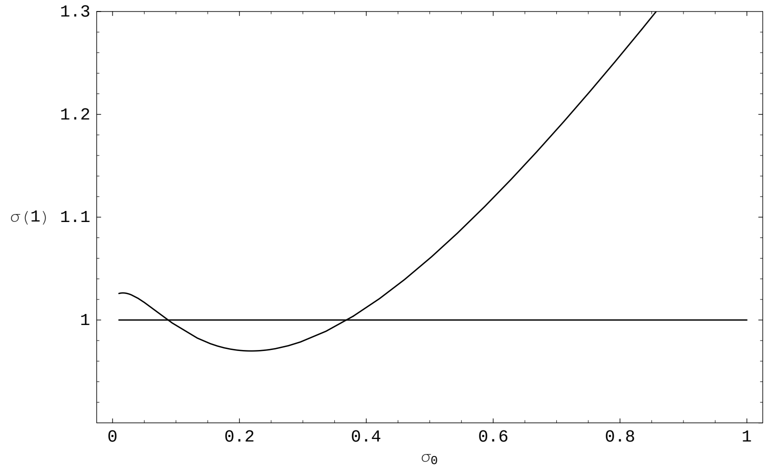

Figure 1: Looking for the maximal solution of eqs. (2.12), $\sigma(1) = 1$, as a function of the initial value $\sigma(0) = \sigma_0$ (in the specific case $d = 1$, $R/L = 0.46$)

4. Numerical solutions and critical lines

Solving eqs. (2.9) serves two purposes: a) to determine a critical line which separates the dispersive (D) and black-hole (BH) phases, and b) to obtain the actual solution, for a given source parameters, and to calculate various on shell observables. Let us illustrate the first procedure in the case of the scattering of the two identical beams with the sources (3.1). There are two independent parameters: d - specifies the shape of the energy distribution, and R/L the ratio of the Schwarzschild radius of a source to its maximal extent L. We shall work in units of L. As already outlined in section 2.1 we would like to find an initial value σ_0, in (2.13), such that the corresponding solution satisfies $\sigma(1) = 1$, matching the $t > L^2$ behaviour (2.11). We shall refer to such a solution as the maximal solution.

To this end, we examine the dependence $\sigma(1)$ on σ_0, at given (d, R), cf. figure 1, and identify the region, in (d, R), where the equation $\sigma(1, \sigma_0) = 1$ has a real solution. It turns out that when we increase R, at fixed d, the minimum in figure 1, moves upwards giving rise to three cases: i) for $R < R_1$ there is only one solution, ii) for $R_1 < R < R_c$ there are two (real) solutions, and iii) for $R_c < R$ there are no real solutions. We then determine the critical values R_c for a range of d thus obtaining the critical line $R_c(d) \equiv (R/L)_c(d)$ in the $(d, R/L)$ space. All numerics is done with the aid of Mathematica.

The final result, cf. figure 2, confirms well our expectations. In the limit of the homogeneous sources ($d = 1$) we recover $R_c = 0.47$ — the value already quoted in [1]. For $d \longrightarrow \infty$ one expects to reach the kinematics of the particle-ring scattering, with $R_c = 0.62$, and indeed the critical line is consistent with this prediction. In figure 2 we also plot the upper bound on R_c coming from the CTS criterion (2.14). It is easy to check that this reads:

$$R_c^{\rm CTS} = (4d(1-d))^{1/4}\,\Theta(1/2 - d) + \Theta(d - 1/2)\,. \tag{4.1}$$

We have also checked that, to a good approximation, our R_c roughly varies as $d^{1/4}$ for small d in nice (though not necessary) agreement with (4.1).

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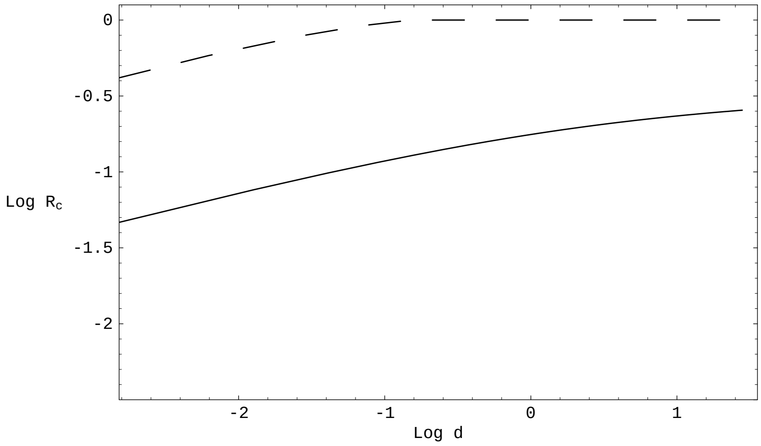

Figure 2: The critical (solid) line in the (R, d) plane having set $L = 1$. We also show (dashed line) the upper bound on R_c from the CTS criterion (2.14). The BH phase is above the solid line.

As a second example consider the scattering of two Gaussian sources, concentrated at $\boldsymbol{x} = 0$ with transverse sizes L_1, L_2, as in (3.8). Strictly speaking, the sources do not have a finite support, however they vanish quickly for $t \gg L_i$. Therefore we replace the condition $\sigma(1) = 1$ with $\sigma(\#Max(L_1, L_2)) = 1$. In practice taking $\# = 10$ is more than sufficient. This time we set $R = 1$ and examine the transition in the (L_1, L_2) plane. The result is shown in figure 3. Obviously, the critical line is symmetric with respect to $L_1 \leftrightarrow L_2$. It's shape corresponds roughly to a straight line $L_1 + L_2 = const$ (as clearly visible in a linear plot not shown here), suggesting that this variable controls the effective concentration of the total energy. Once more we can ask how this critical line compares with the lower bound one would find using the CTS criterion (2.14). This can be easily done numerically and the result is shown again in figure 3. Amusingly, also the CTS criterion gives a curve which, in the central region $L_1 \sim L_2$, roughly corresponds to $L_1 + L_2 = const$. Besides a typical factor 2 discrepancy (but remember: CTS criteria only give bounds!), the two curves start to diverge when the two L_i are very different. In fact, while the CTS-curve goes to the point $L_1 = 0, L_2 = \frac{1}{\sqrt{2}}$, our result indicates that this point moves to infinity as $L_1 \to 0$, i.e. that one is always in the BH regime in that limit.

All the above calculations required solving eqs. (2.12) for $t \leq 1$. Next we look for the maximal solution in the larger domain $t \leq t_{\max}$. To this end it suffices to use the value σ_0 found above and extend the profile $R_i(t)$ beyond L^2: $R_i(t) = R, t > L^2$. The rest is again done by Mathematica. As an example, we show in figure 4 the derivative of the solution $\rho(t)$. It interpolates smoothly around $t = 1$ and, as expected, tends to 1 at large t. We have also verified that for $t > 1$, $\rho(t)$ obtained above is identical to the analytic solution for the constant profiles

$$\rho_{an}(t) = R^2 F^{-1} \left[F(\rho_0/R^2) + t/R^2 - t_0/R^2 \right], \tag{4.2}$$

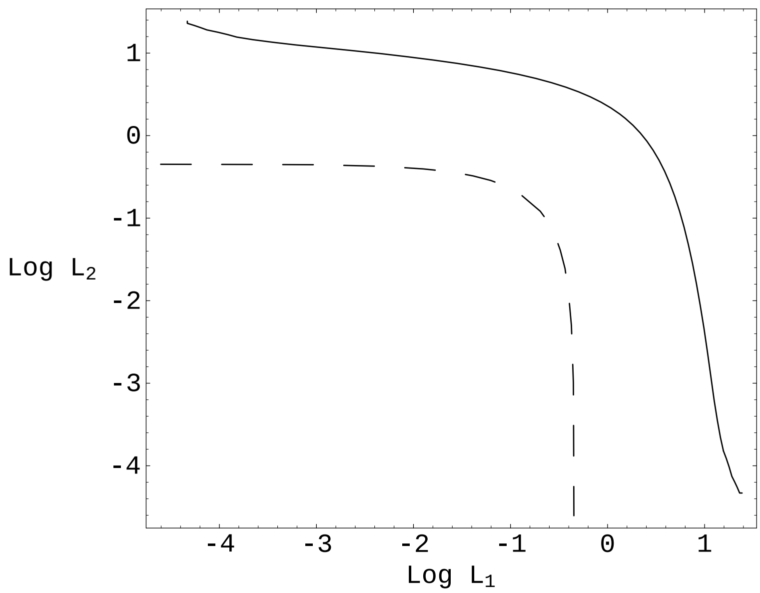

Figure 3: Two gaussian sources around the origin. The BH phase lies below the critical (solid) line in the (L_1, L_2) plane. We also show (dashed line) the lower bound on the curve from the CTS criterion (2.14).

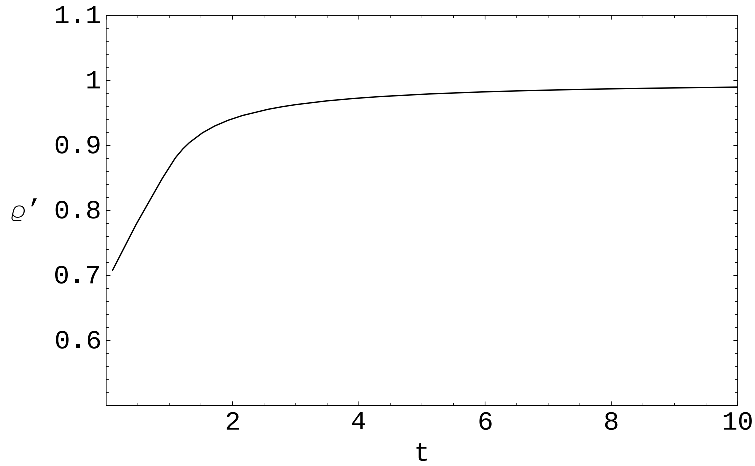

Figure 4: The maximal solution, in the dispersive phase, also for $t > 1$ (two identical sources, eq. (3.1), with $d = 1$, $R/L = 0.44$).

with $\rho_0 = \rho(t_0), \quad t_0 > 1$, and

$$F(x) = \sqrt{x(x-1)} + log(\sqrt{x} + \sqrt{x-1}). \tag{4.3}$$

5. Behaviour near the critical point

An interesting quantity to compute is obviously the on-shell action, i.e. the value of the action on the e.o.m. as a function of the external parameters (the profiles of the external

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sources). In the multidimensional parameter space we expect to find, as in the collapse problem, a critical surface of co-dimension 1 where the on-shell action itself develops some branch-point singularity. Typically, on the other side of that critical surface the action acquires an additional imaginary part (on top of the one due to graviton emission) providing an additional absorption of the elastic amplitude [1].

This extra absorption could possibly be interpreted as the opening-up of new channels e.g. those corresponding to the formation of black holes behaving as stable particles at leading order in $\hbar$. The absorption itself is expected to be related (up to some numerical factor) to the number of new channels that are opening up beyond criticality and to provide therefore information on the entropy of the black holes that are supposedly formed. Under these assumptions a connection can possibly be established between the critical exponents of the action and those of Choptuik's scaling [10].

On the equations of motion the action (2.8) immediately simplifies to read:

$$\frac{\mathcal{A}^{eom}}{\hbar} = -\frac{1}{4l_P^2}\int dt \left[(1-\dot{\rho})^2 - \frac{1}{\rho}R_1(t)R_2(t)\right] , \tag{5.1}$$

while using the second of eq (2.9) the above expression can also be written in the form:

$$\begin{aligned}\frac{\mathcal{A}^{eom}}{\hbar} &= -\frac{1}{4l_P^2}\int dt \left[(1-\dot{\rho})^2 - 2\rho\ddot{\rho}\right] \\ &= -\frac{1}{4l_P^2}\int dt \left[3(1-\dot{\rho})^2 - 2t\ddot{\rho} - 2\frac{d}{dt}[(t-\rho)(1-\dot{\rho})]\right] \\ &= -\frac{1}{4l_P^2}\int dt \left[3(1-\dot{\rho})^2 - 2t\ddot{\rho}\right] .\end{aligned} \tag{5.2}$$

Here we have used the fact that the last surface term vanishes because of the properties of the solution at $t=0$ and $t=\infty$.

We would like to study the action near the critical point where we expect to find a branch point behaviour. Unfortunately, the action itself is IR-singular: the infinite Coulomb phase, even if it is unobservable, has to be subtracted carefully in a numerical analysis in order to avoid that interesting finite terms are subtracted as well. Also, the branch point appears to be quite soft: a power $(p-p*)^{3/2}$ (p being a generic source parameter with $p*$ its critical value) behaviour which has to be unraveled from leading analytic pieces (a constant and a linear term).

Fortunately, a trick can be found in order to avoid the IR-singularity. Consider the partial derivative of the on-shell action wrt a particular parameter of the problem, e.g. the total cm energy. In general, this derivative will receive two contributions: one from the *explicit* dependence of the action from that parameter; and one from the *implicit* dependence of the solution (the fields) upon the parameter. However, this second contribution is also proportional to the variation of the action with respect to the fields which, by definition of the classical equations, vanishes on shell. We are thus led to:

$$\frac{\partial \mathcal{A}}{\partial p} = \int \left(\frac{\partial L}{\partial p}\right)_{\text{fixed fields}} . \tag{5.3}$$

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We can use this general strategy to our advantage in the case at hand by considering derivatives of $\frac{\mathcal{A}}{Gs}$ and, as parameter p, the overall cm energy R^2 while keeping the shapes of the two sources, $R_i(t)/R$ fixed. Since:

$$\frac{\mathcal{A}}{Gs} = -\int dt \left[\frac{(1-\dot\rho)^2}{R^2} - \frac{R_1(t)R_2(t)}{R^2\rho} + (2\pi)^2\rho\left(\dot a_1 + \frac{R_1(t)}{2\pi R\rho}\right)\left(\dot a_2 + \frac{R_2(t)}{2\pi R\rho}\right)\right], \quad (5.4)$$

we see that only the first term contributes to the abovementioned partial derivative so that:

$$\frac{\partial(\mathcal{A}/Gs)}{\partial R^2} = \frac{1}{R^4}\int dt(1-\dot\rho)^2\,, \quad (5.5)$$

which is now perfectly IR-finite.

Interestingly, the above derivative of the action is directly related to the average multiplicity of emitted gravitons. Using the effective form of the complex action introduced in [1] one readily obtains:

$$\frac{\partial(\mathcal{A}/Gs)}{\partial R^2} = \frac{\pi^2}{R^3\sqrt{s}}\langle N\rangle\,, \quad (5.6)$$

which shows that the total multiplicity is in fact IR-finite in the axisymmetric case (recall that the IR-sensitive polarization is not produced in this case).

We are now ready to analyze numerically the behaviour of the on shell action in the vicinity of the critical point. In figure 5 we show $R^4\partial_{R^2}(\mathcal{A}/Gs)$, as obtained from the integral (5.5) in a range of R's. At first sight one might suspect that $\langle N\rangle$ is infinite exactly at R_c. A careful analysis shows that it is instead finite although it approaches the critical point with an infinite derivative. The fit, shown by a solid line, is consistent with a square root branch point,

$$\langle N\rangle = c_0 + c_1(R_c - R)^{\frac{1}{2}}\,, \quad (5.7)$$

in the vicinity of R_c. This confirms the validity of the expansion

$$\mathcal{A}(R) = A_0 + A_1(R_c - R)^b + A_2(R_c - R)^c\,, \quad (5.8)$$

with $b = 1$ and $c = 3/2$. Such a behaviour, first found in [1] and later confirmed in [7], could be possibly related [1] to a Choptuik-like [10] dependence of the mass of the barely formed black hole from $(R - R_c)$. In that case our result would be interpreted as meaning that $M_{\mathrm{BH}} \sim (R - R_c)^\gamma$, with $\gamma = 3/4$, an exponent which is about twice the one found by Choptuik [10] for the spherical collapse of some fluids. It will be very interesting to see what values of γ will come out of future numerical studies of the collapse of extended sources of the type discussed here.

One can also check directly the behaviour of the full, suitably regularized, action (5.1) close to the transition point. We have used two regularizations: 1) the integral was cutoff at some large value $t_{\max}$, defining the cutoff action $\mathcal{A}_{\mathrm{cut}}$, and 2) we have introduced a counterterm, $1/\rho(t) \longrightarrow 1/\rho(t) - 1/t$, which subtracts the infrared divergence and results in the IR and UV-finite expression $\mathcal{A}_{\mathrm{sub}}$. Figure 6 shows these actions together with fits like (5.8) and two *a priori* unconstrained powers.

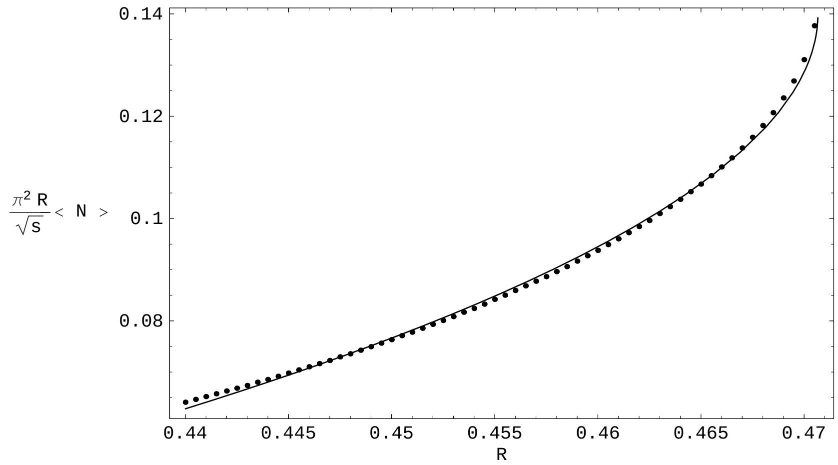

Figure 5: The total multiplicity of emitted gravitons (points) and the best fit: $0.138 - 0.46(R_c - R)^{0.523}$. A fit with the fixed power 1/2 is marginally worse.

$t_{\max}$	b	A_0	A_1/A_0	A_2/A_1
20	1.044	0.826	−6.149	−0.894
50	1.045	1.034	−6.096	−0.883
100	1.046	1.189	−6.077	−0.877
300	1.049	1.433	−6.060	−0.871

Table 1: Parameters of the fit (5.9) for a range of cut-offs, $t_{\max}$

Comparing the two plots one sees immediately that the cutoff action is much more linear than the subtracted one. What happens is that $\mathcal{A}_{\rm cut}$ is obviously much bigger (cf. the scales of both figures) and the nonlinear effect of the last term is relatively weaker. On the other hand, in $\mathcal{A}_{\rm sub}$ the big "background" is already subtracted exposing better the last two terms. This observation is also confirmed by our fits. Simple Mathematica fitting routines were unable to disentangle two different powers in $\mathcal{A}_{\rm cut}$, while fitting $\mathcal{A}_{\rm sub}$ gave $b = 1.07 \pm 0.12$ and $c = 1.34 \pm 0.22$ in good agreement with $b = 1$ and $c = 1.5$ predicted in [1]. As for $\mathcal{A}_{\rm cut}$, we were glad to find that a slightly less general form,

$$\mathcal{A}(R) = A_0 + A_1(R_c - R)^b + A_2(R_c - R)^{b+1/2}\,, \tag{5.9}$$

works well, cf. figure 6 (upper part). Finally, we have also checked the cutoff dependence of these parameters, see table 1.

As expected the power b is cutoff independent and close to the predicted value $b = 1$. The amplitudes A_i show a mild, possibly logarithmic, dependence on $t_{\max}$. Interestingly, the ratios of these amplitudes seem to depend very weakly on $t_{\max}$ suggesting a possible factorization of the IR divergent contributions.

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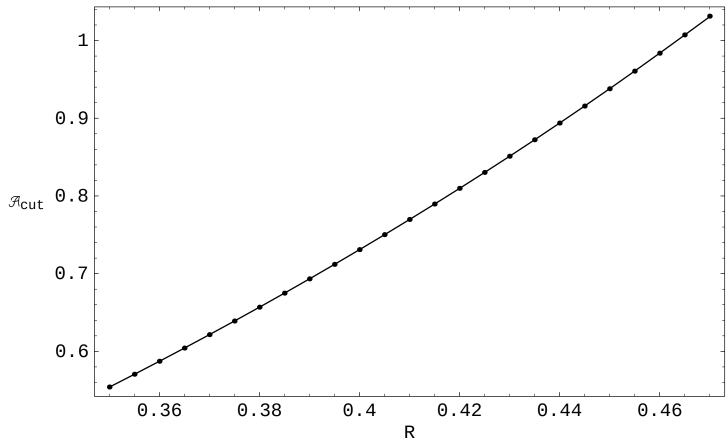

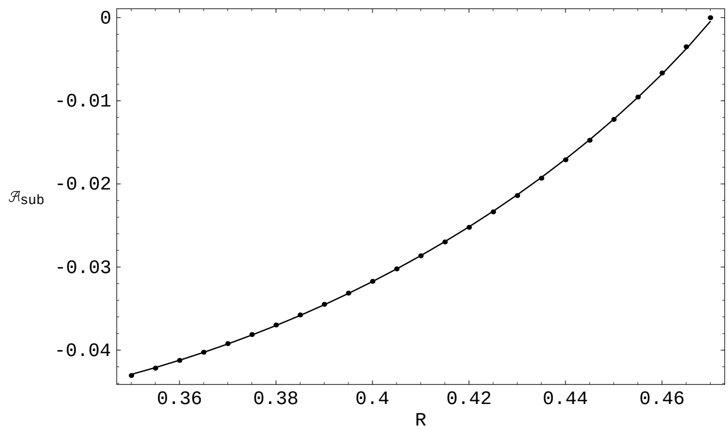

Figure 6: Fitting the critical behaviour of the cutoff (upper, $t_{\max} = 50$) and the subtracted actions.

6. Summary and future prospects

In this paper we have continued the ambitious program initiated in ACV [1] by focussing on the case of axisymmetric collisions. Both the ACV approach and the collapse criteria based on the explicit construction of minimal CTS [5, 6] greatly simplify in this case. In the former the axial symmetry removes the infrared problem and reduces the equations to ordinary differential equations, often allowing for quasi-analytical solutions. The CTS criteria also take a much simpler form, as exemplified by eq. (2.14).

In spite of its simplicity, this case becomes extremely rich if one generalizes the problem of particle-particle scattering to the one of the central collisions of two extended sources. The latter problem depends, in principle, on an infinite number of parameters (describing the energy profile of each beam) and thus allows to study, in principle, how the information about the initial state is transmitted to the final state.

In this first paper we have concentrated our attention on the relevant system of equations in position space, on the determination of the critical points/surfaces, and on the comparison with CTS-based results. Our results have confirmed a rather striking connection between CTS collapse criteria and the existence of critical points (in general hypersurfaces) in the parameter space describing the two incoming sources. Not only could we prove a general theorem that the CTS criterion of [6] implies the absence of real regular solutions of the ACV equations; we have also found that the bounds on the critical values based on [6] are typically only a factor two off the ACV predictions.

We have also investigated the behaviour of the on-shell action, hence of the S-matrix, as one approaches the critical points, confirming the robustness of a definite fractional power-law behaviour for the phase of the S-matrix. Within some assumptions this might be interpreted as a critical "Choptuik exponent" [10] for the behaviour of the mass of the black hole as one approaches the critical point from the collapse side.

Unfortunately, techniques for going beyond the critical points are still rudimentary, while the interpretation of the extra absorption of the elastic amplitude as being due to the opening of new (black-hole formation?) channels remains very speculative. A full understanding of the role of inelasticity (i.e. of graviton emission) even in the subcritical regime is still necessary and might shed light of the nature of the "phase transition" between a "full dispersion" regime and one in which a fraction of the incoming energy collapses. For all these questions a momentum-space approach (see again [1]) — rather than the one in position space adopted here — may turn out to be more suitable. We plan to present it in a near-future investigation.

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